The eigenstructure representation of groundwater dynamics, as a precursor for aquifer management

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Warning

The following presentation contains mathematical equations.

Some viewers may experience periods of confusion and boredom.

Summary

1. The aquifer management problem

2. Groundwater dynamics – ARMAX serendipity

3. Eigenstructure of groundwater flow equation

4. Eigenstructure of numerical groundwater flow model

5. Analytical solutions for simple groundwater systems

6. Application of analytical solutions to aquifer management

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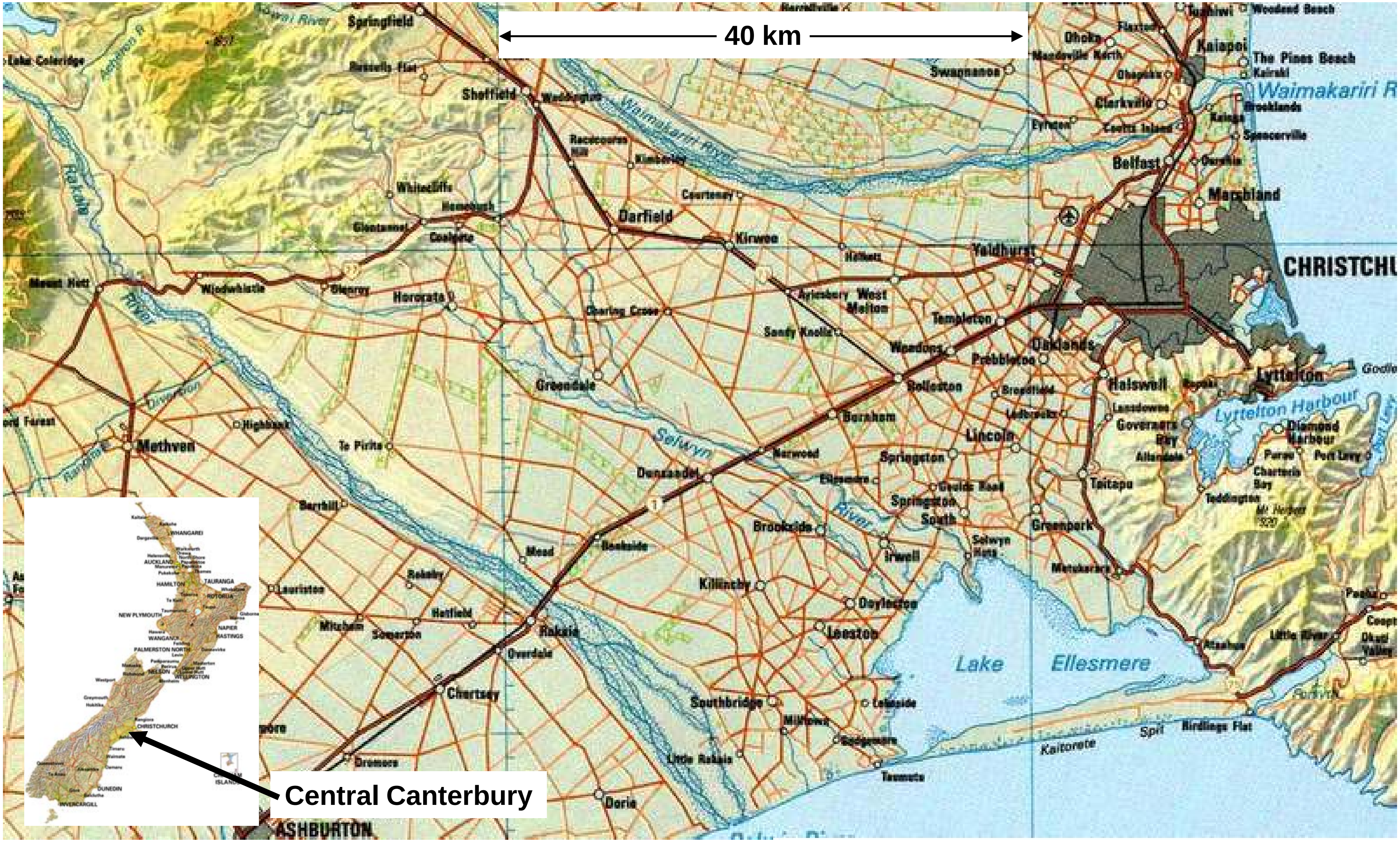
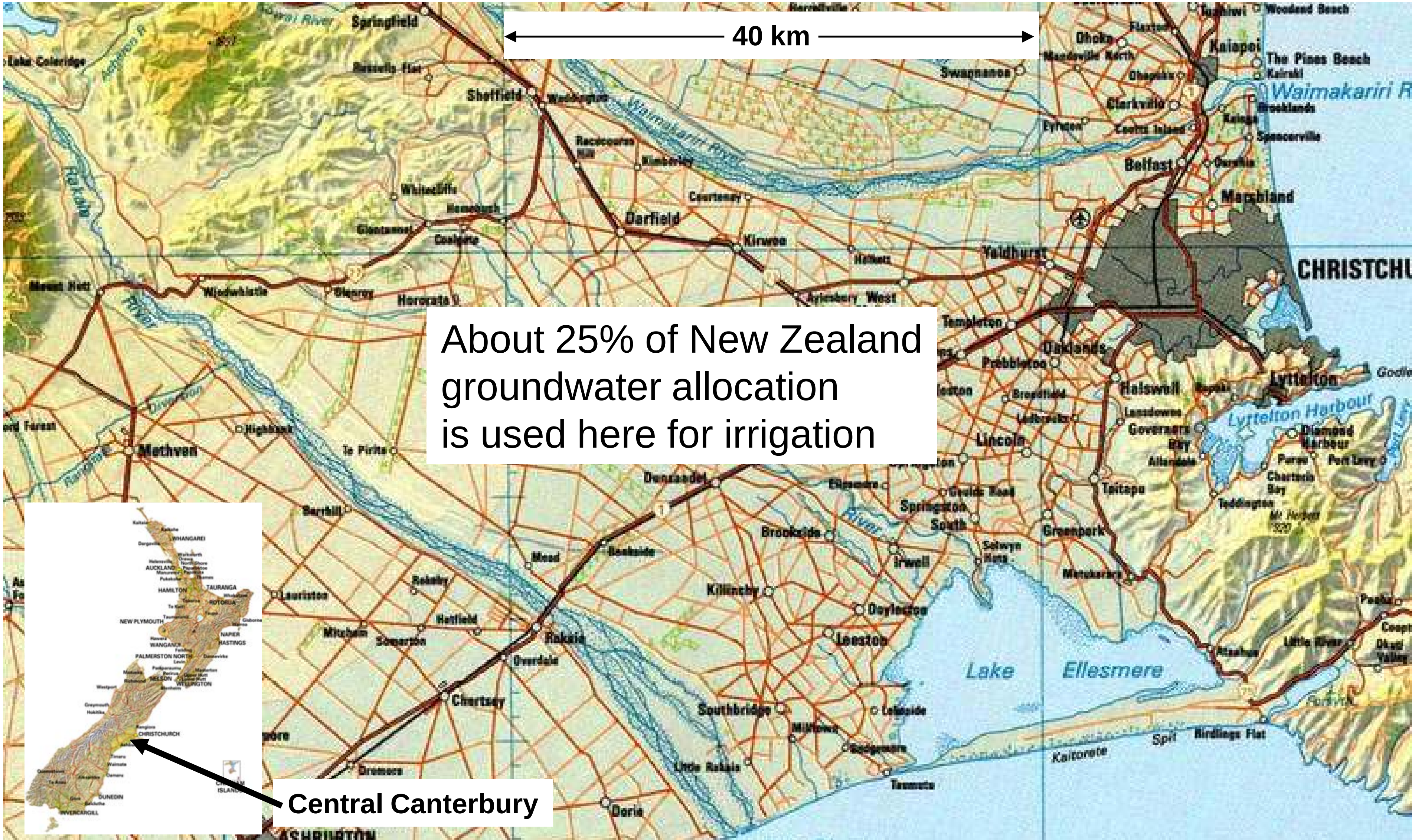
Canterbury Plains, New Zealand

Image Science and Analysis Laboratory, NASA-Johnson Space Center

Central Canterbury



Central Canterbury

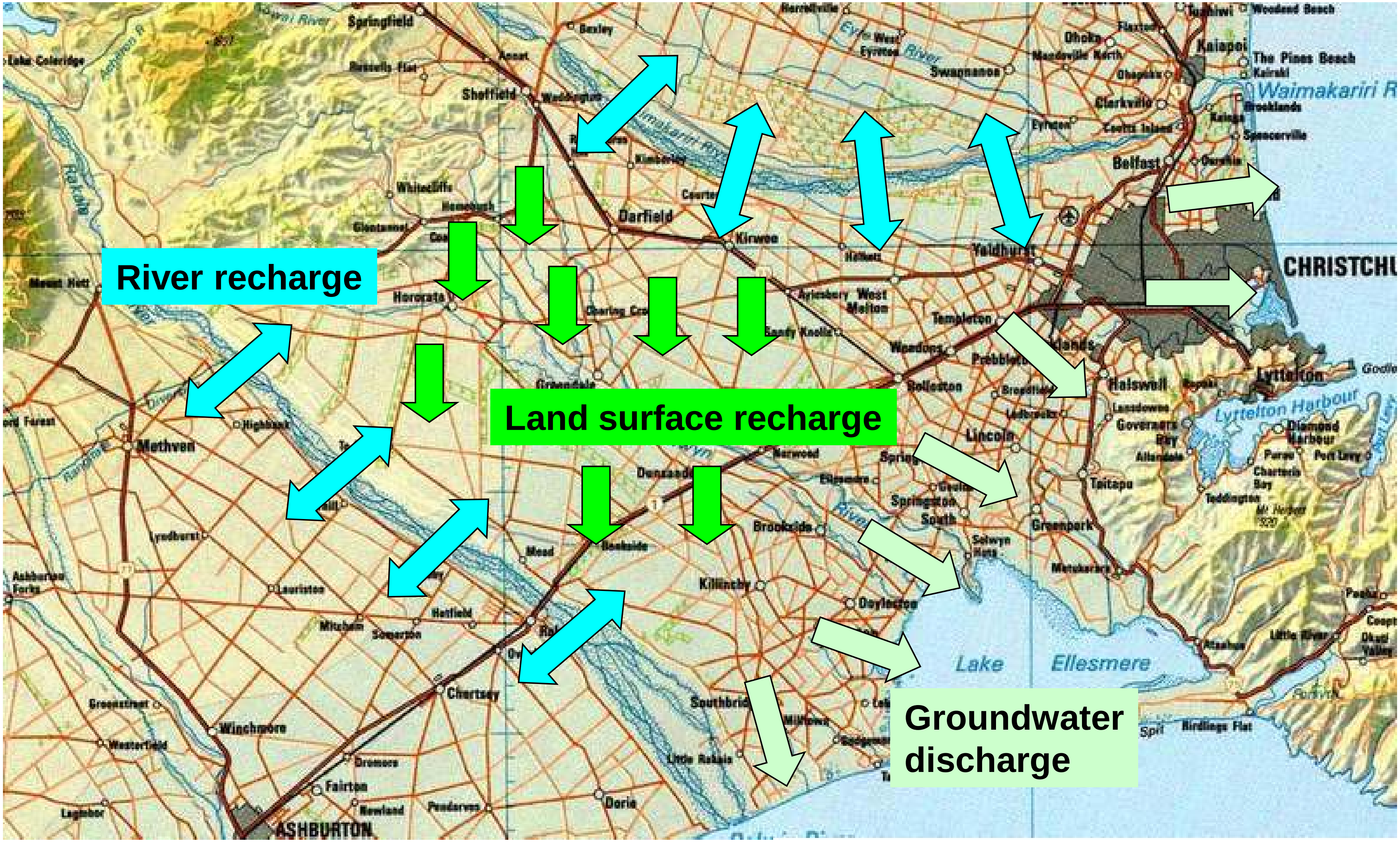


About 25% of New Zealand groundwater allocation
is used here for irrigation

Central Canterbury Aquifer System

Alluvial aquifers
200 – 500 m thick
Area ~ 2300 km^2

Aquifer recharge and discharge

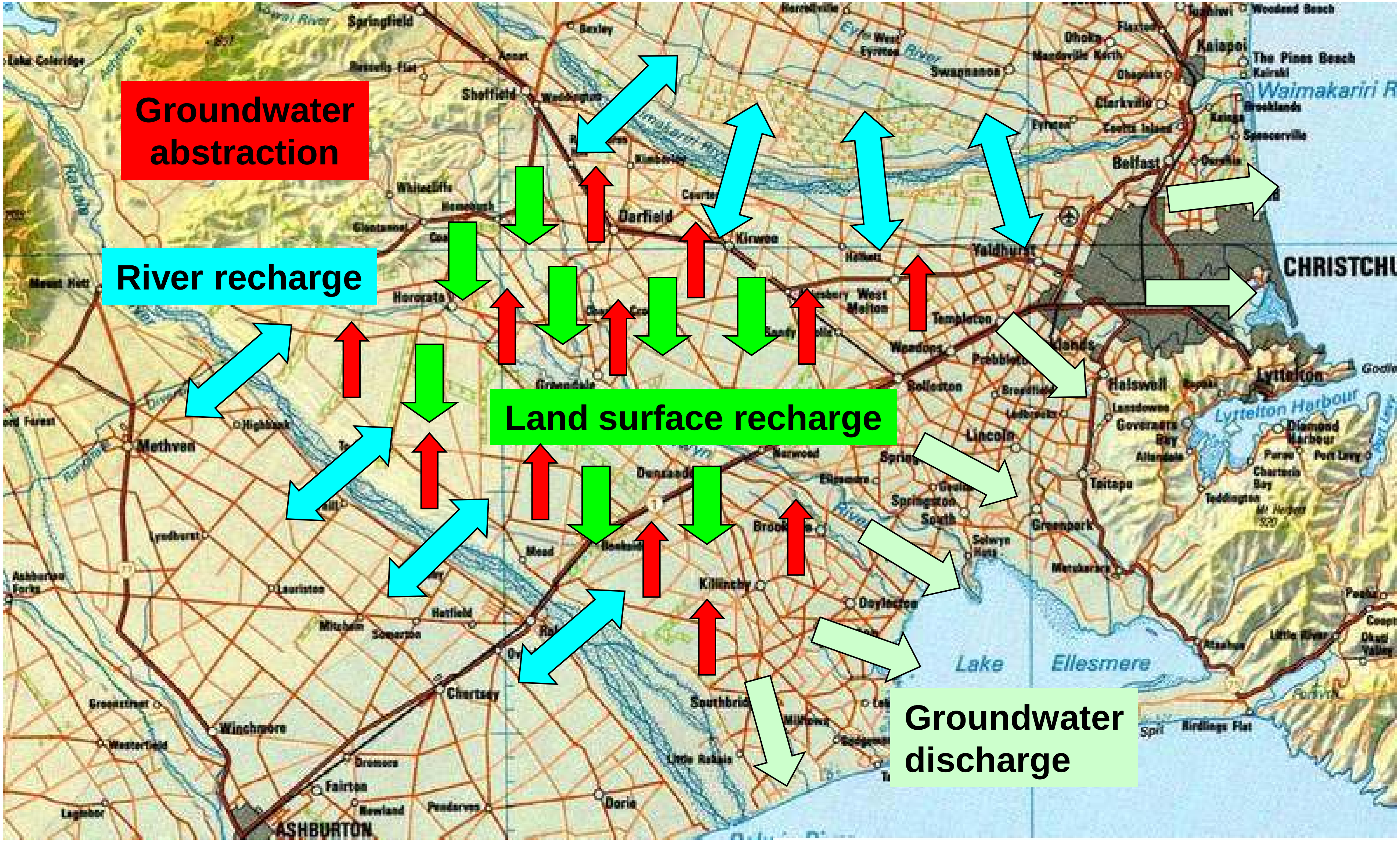


Effect of recharge dynamics on groundwater level (piezometric or hydraulic head)

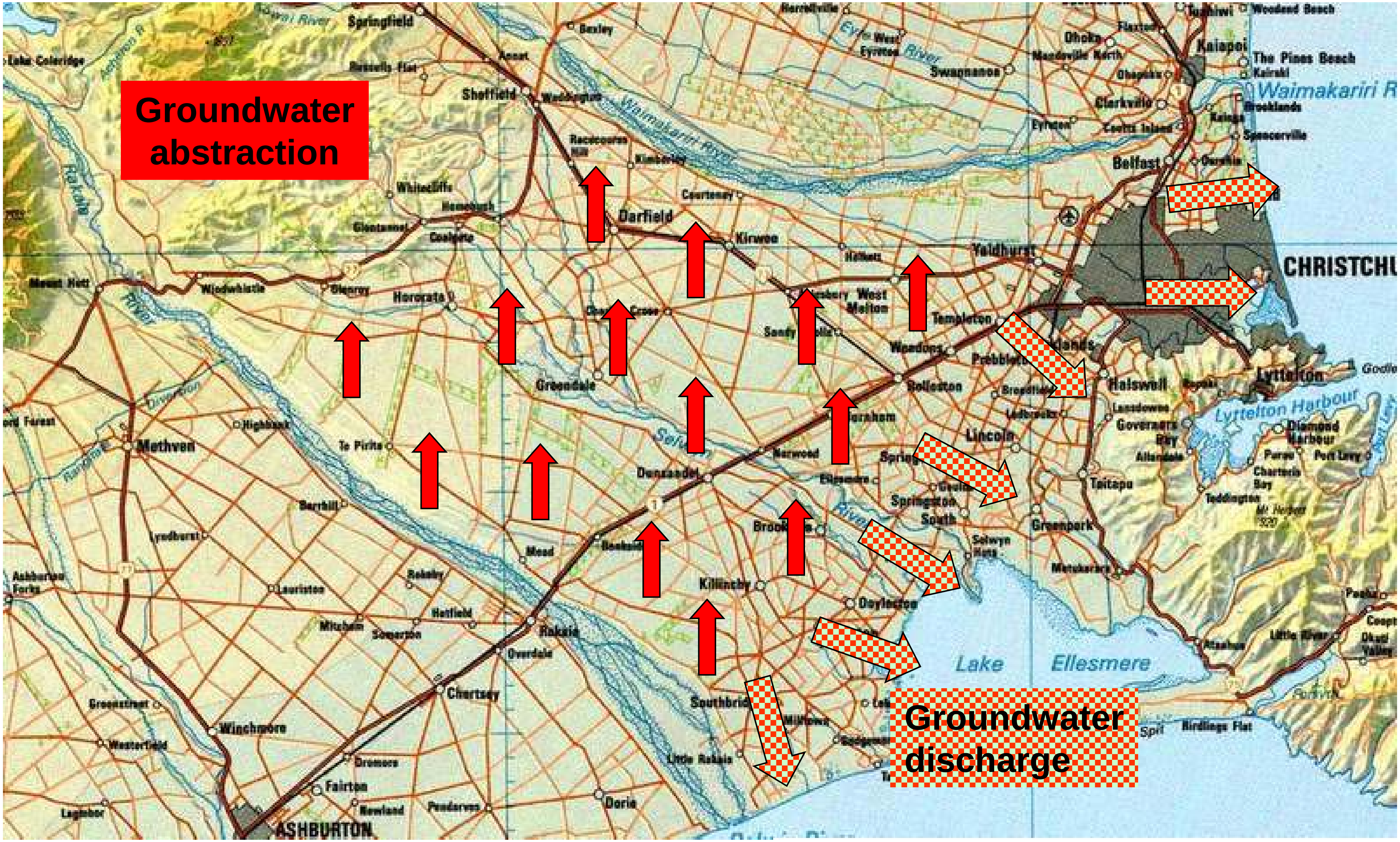
River recharge | **steady piezometric effect (datum)**

Land surface recharge | **causes most of the time variation**

Groundwater abstraction – mainly for irrigation



Groundwater abstraction affects discharge to surface waters



The aquifer management problem

- Balance abstraction for economic benefit against effects on surface water environment (Resource Management Act)

- Allocate groundwater abstraction consents for long term and seasonal allocations

- Seasonal variation to permitted abstraction must take account of agricultural operations

- **Need to understand groundwater dynamics for design of feasible controls**

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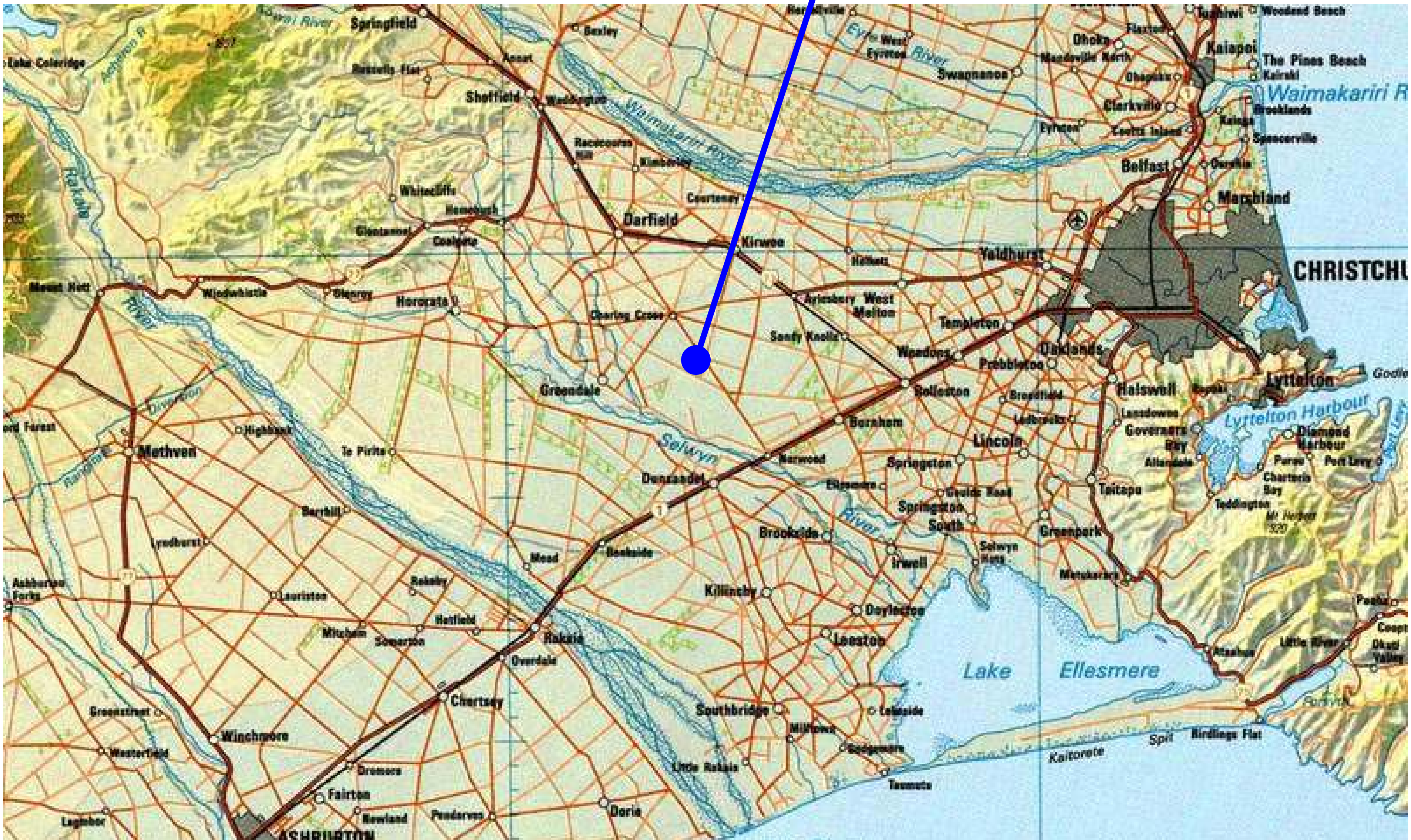
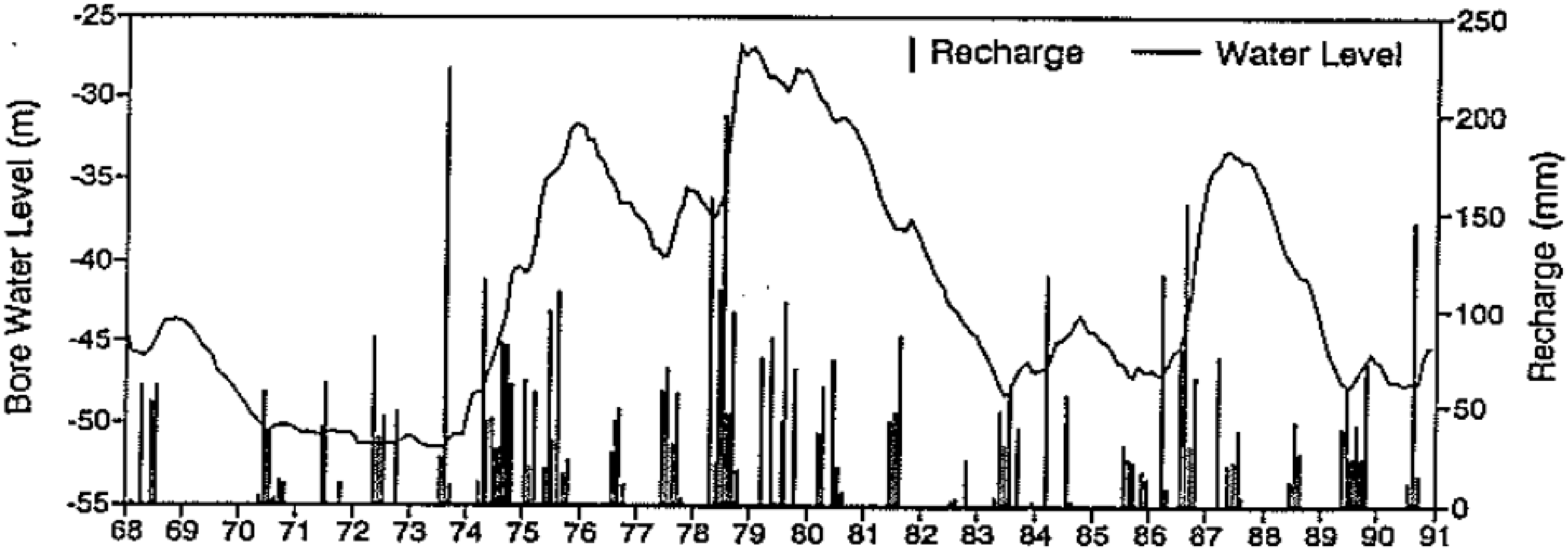
Monitoring well L36/0092



Monthly values of land surface recharge for Central Canterbury Plains and groundwater level at L36/0092



(a)

Bidwell, Callander and Moore (1991)

ARMAX transfer function and noise model - Instrumental Variable method

MICROCAPTAIN software (Young and Benner, 1991)

Observed groundwater level
(deviations from mean)

Land surface recharge
(deviations from mean)

$$H_k = \frac{\mathrm{B}(z^{-1})}{\mathrm{A}(z^{-1})} R_k + \frac{\mathrm{D}(z^{-1})}{\mathrm{C}(z^{-1})} e_k$$

Predicted groundwater level L_k
(deviations from mean)

Noise term

Calibrated transfer function model

Second order polynomial in z^{-1}

$$L_k = \frac{9.79}{\left(1 - 1.705z^{-1} + 0.714z^{-2}\right)} R_{k-1}$$

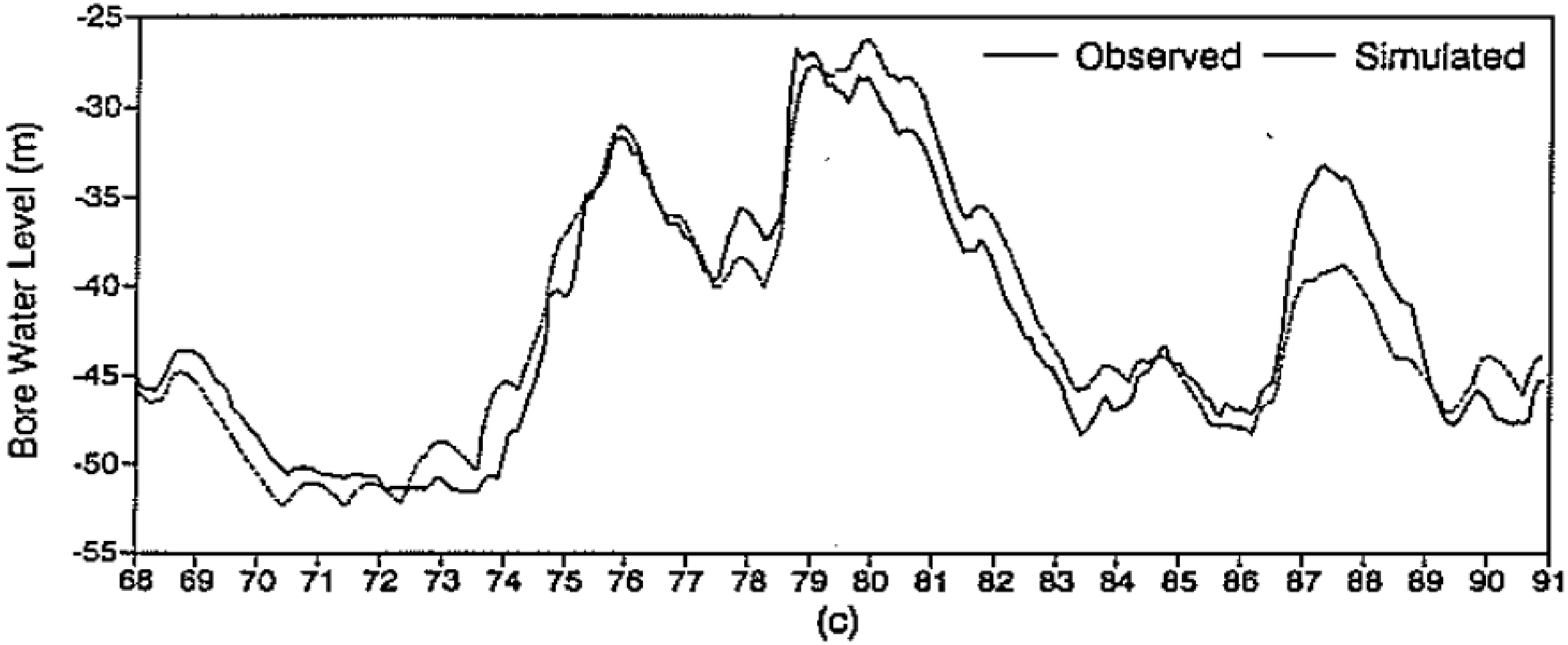
Two real roots (eigenvalues)

$$L_k = \frac{9.79}{\left(1 - 0.965z^{-1}\right)\left(1 - 0.740z^{-1}\right)} R_{k-1}$$

Difference equation

$$L_k = 1.705L_{k-1} - 0.714L_{k-2} + 9.79R_{k-1}$$

ARMAX model prediction of groundwater level at L36/0092 – $R^2 = 0.90$



Bidwell, Callander and Moore (1991)

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Groundwater flow equation
2D-horizontal, heterogeneous, anisotropic, linear

Hydraulic head - *h*

Recharge

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right) + R = S\frac{\partial h}{\partial t}$$

Storativity

Transmissivity tensor

Eigenstructure from Sturm-Liouville problem

Sahuquillo (1983)

eigenfunction　　　　eigenvalue

$$\frac{\partial}{\partial x}\left(T_x \frac{\partial V_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial V_i}{\partial y}\right) + \alpha_i S V_i = 0$$

Eigenfunctions V_i are orthogonal to storativity S

Eigenvalues α_i are real

Groundwater PDE becomes a set of first-order ODE

hydraulic head = transient + steady

$$h(x,y,t) = u(x,y,t) + w(x,y)$$

$$u(x,y,t) = \sum_{i=1}^{\infty} V_i(x,y) L_i(t)$$

system states

eigenfunction

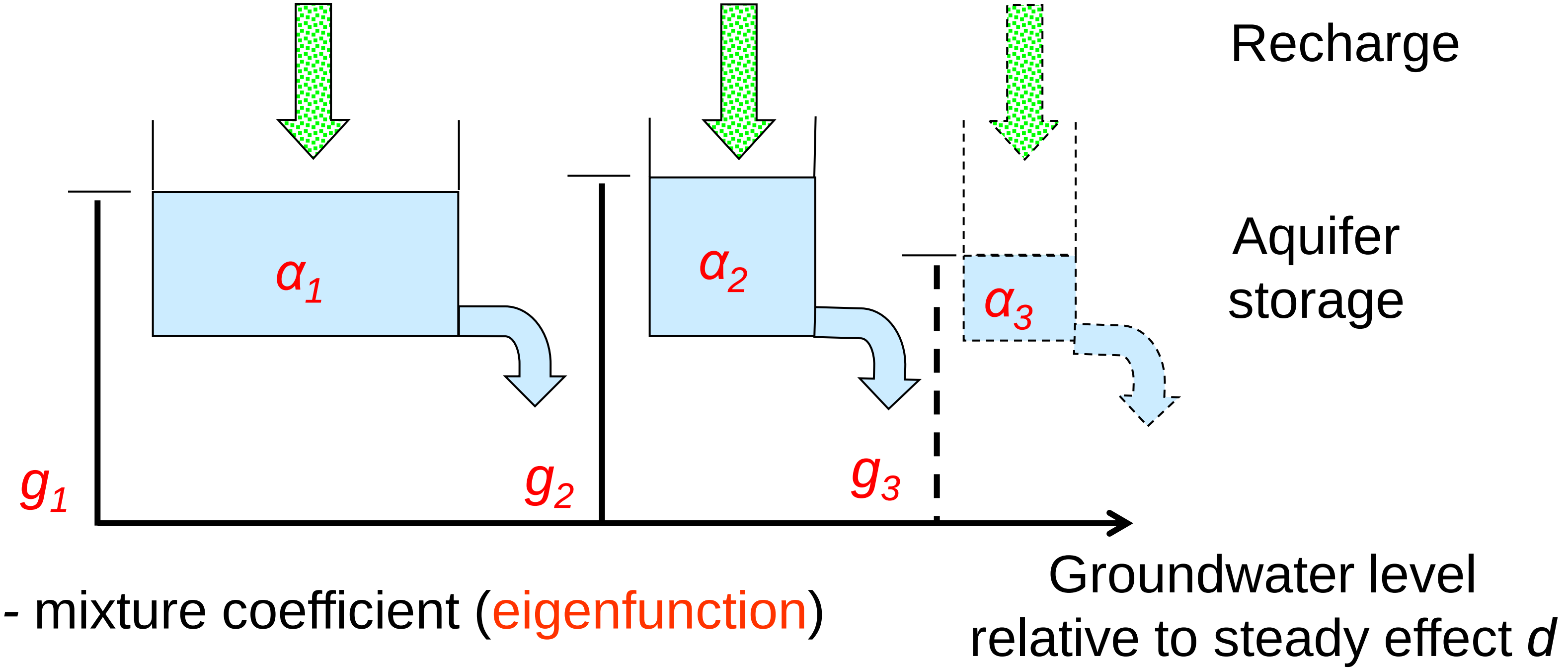
$$\frac{dL_i(t)}{dt} + \alpha_i L_i(t) = c_i(x,y) R(x,y,t)$$

eigenvalue

Eigenstructure as conceptual water storages

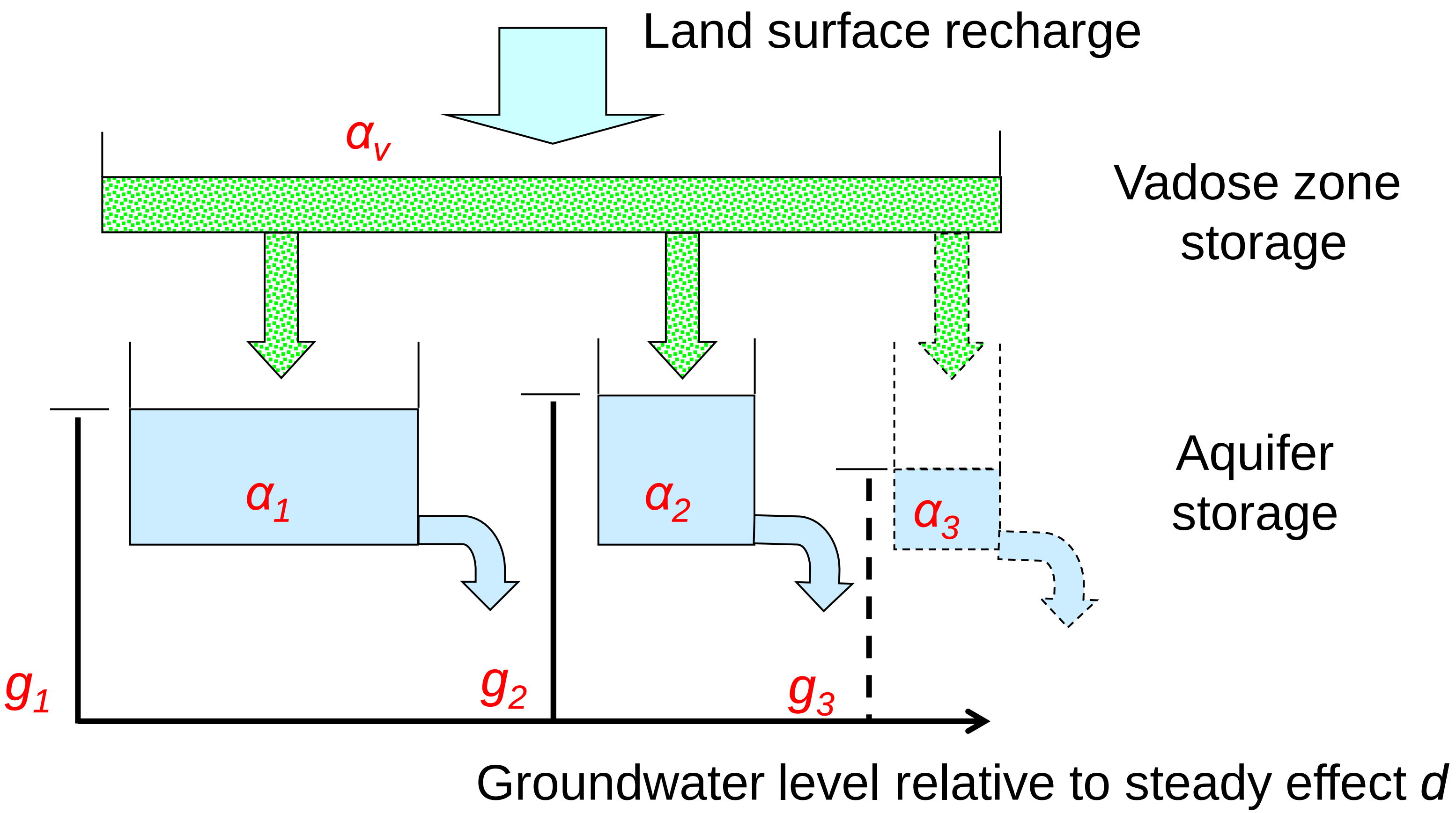
α_i - discharge coefficient (eigenvalue)

discharge = α_i x “storage” states (L_i)

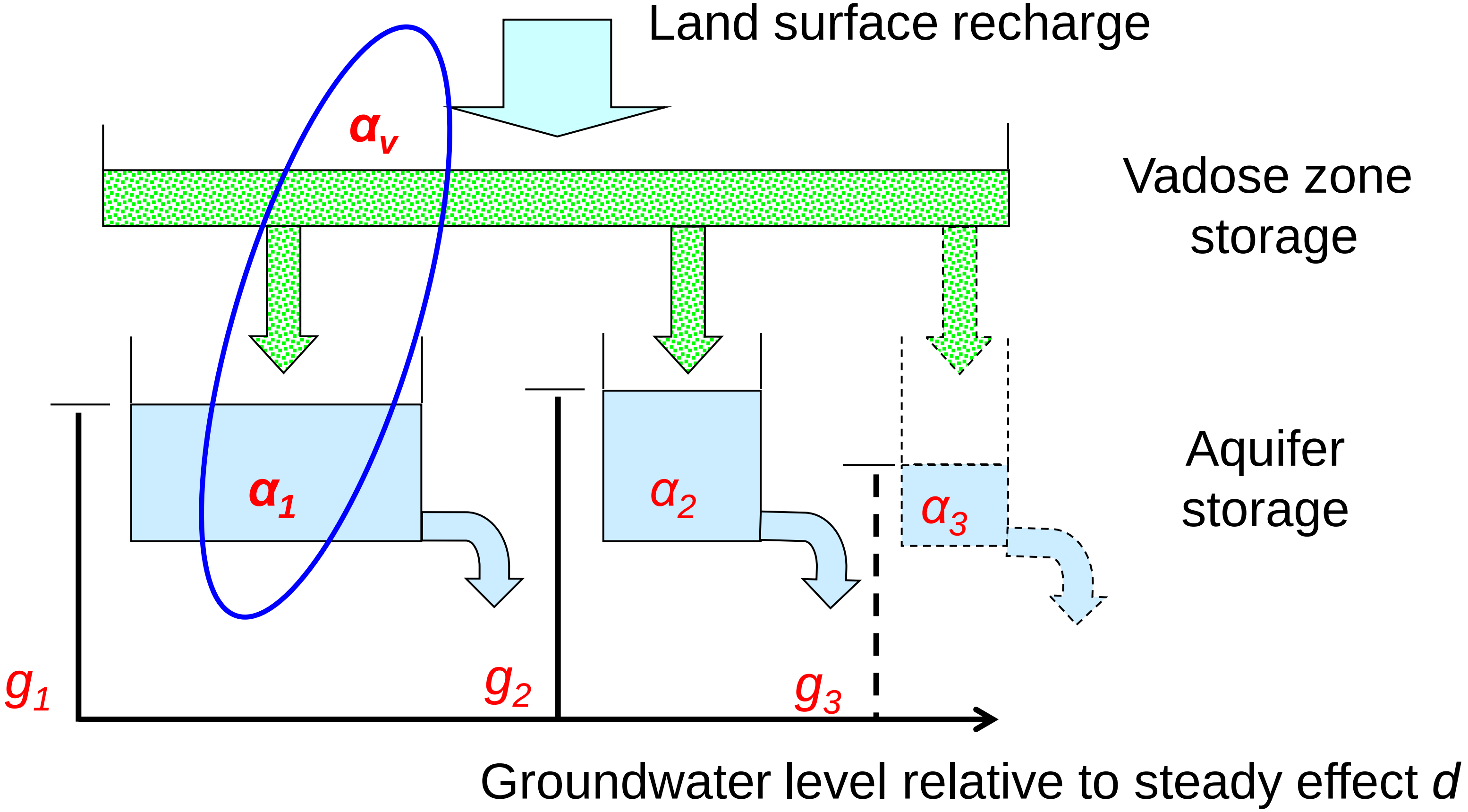


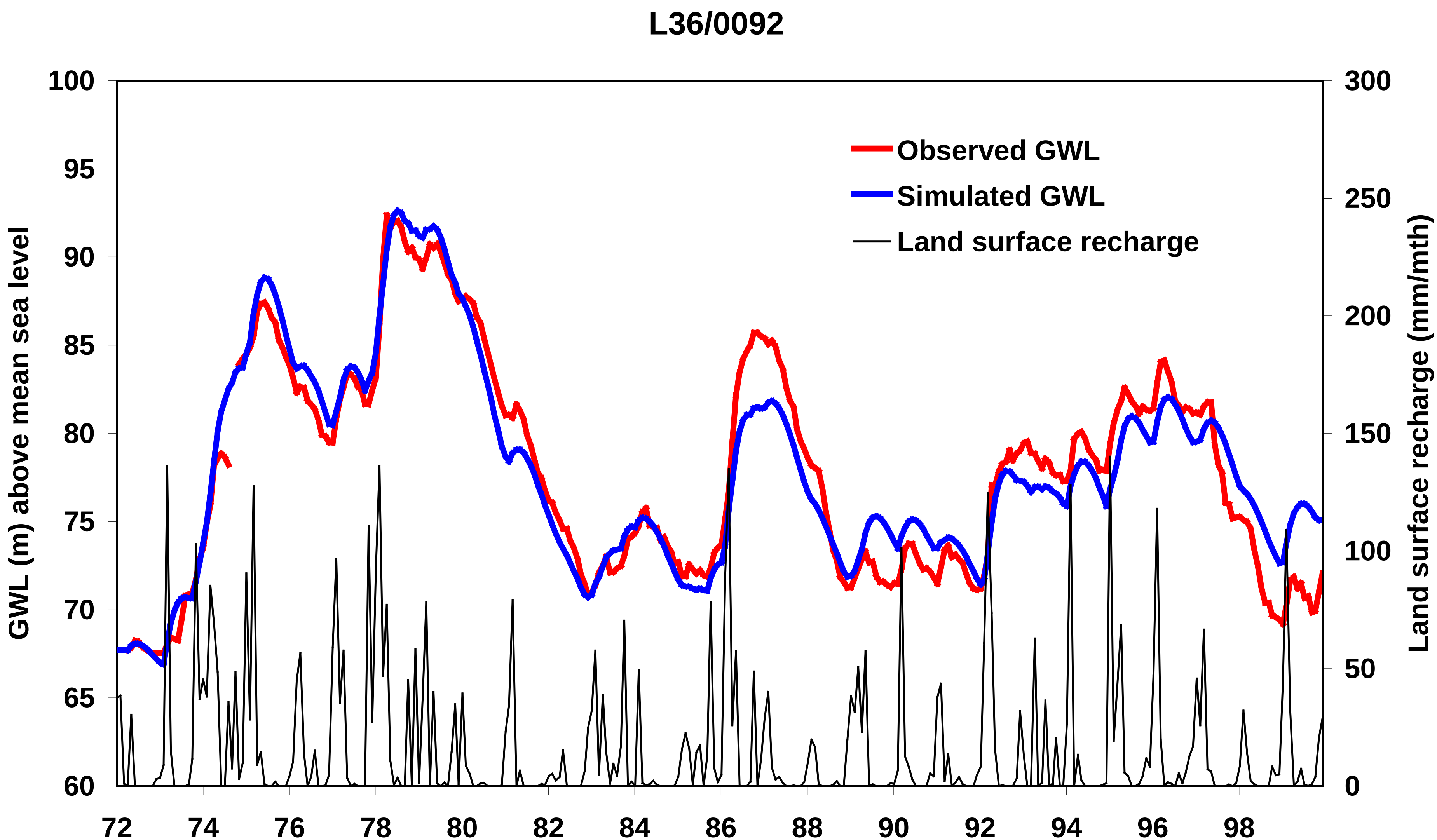
g_i - mixture coefficient (eigenfunction)

Addition of vadose zone storage

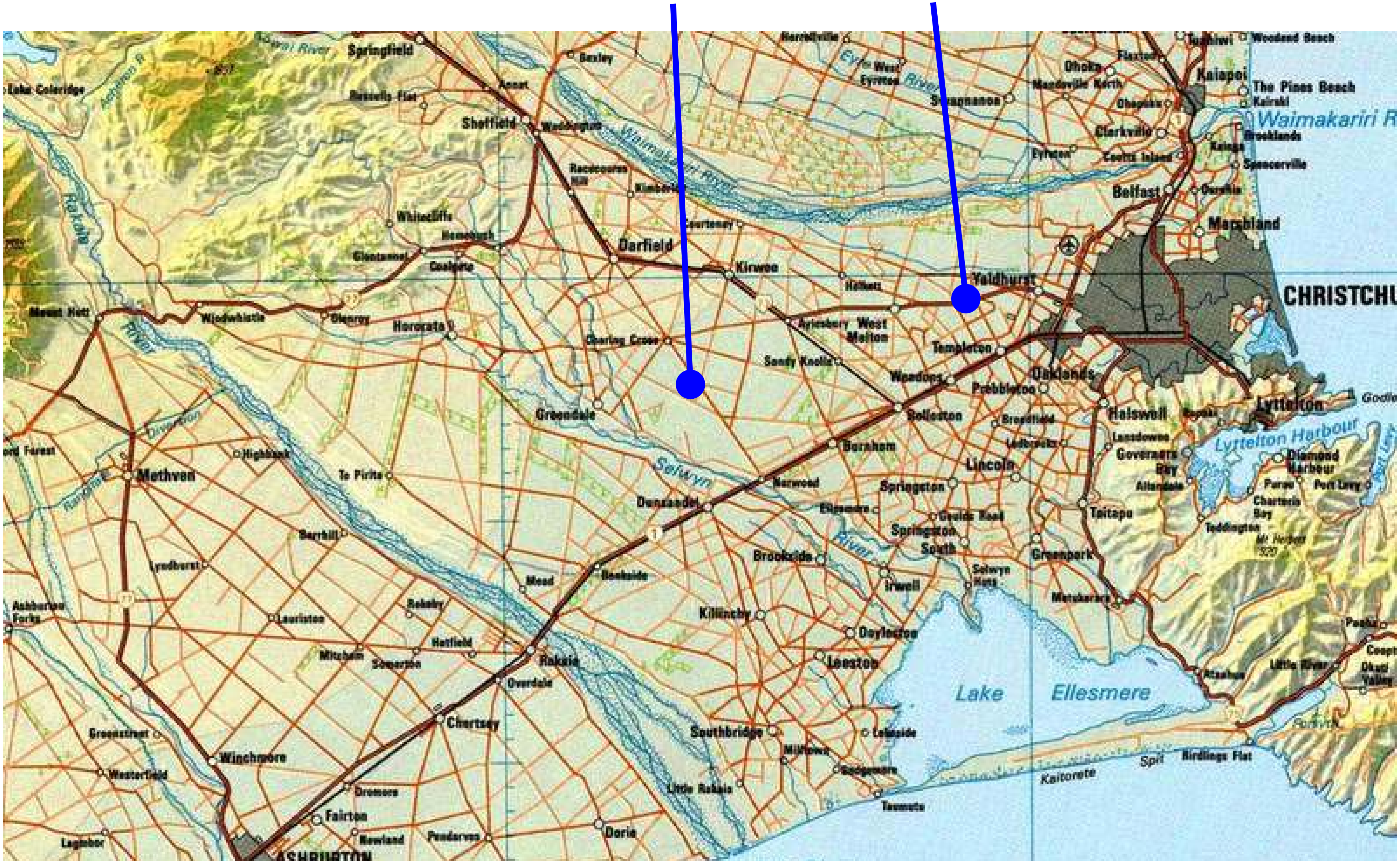


ARMAX identification for L36/0092
deep vadose zone – perched aquifer effect

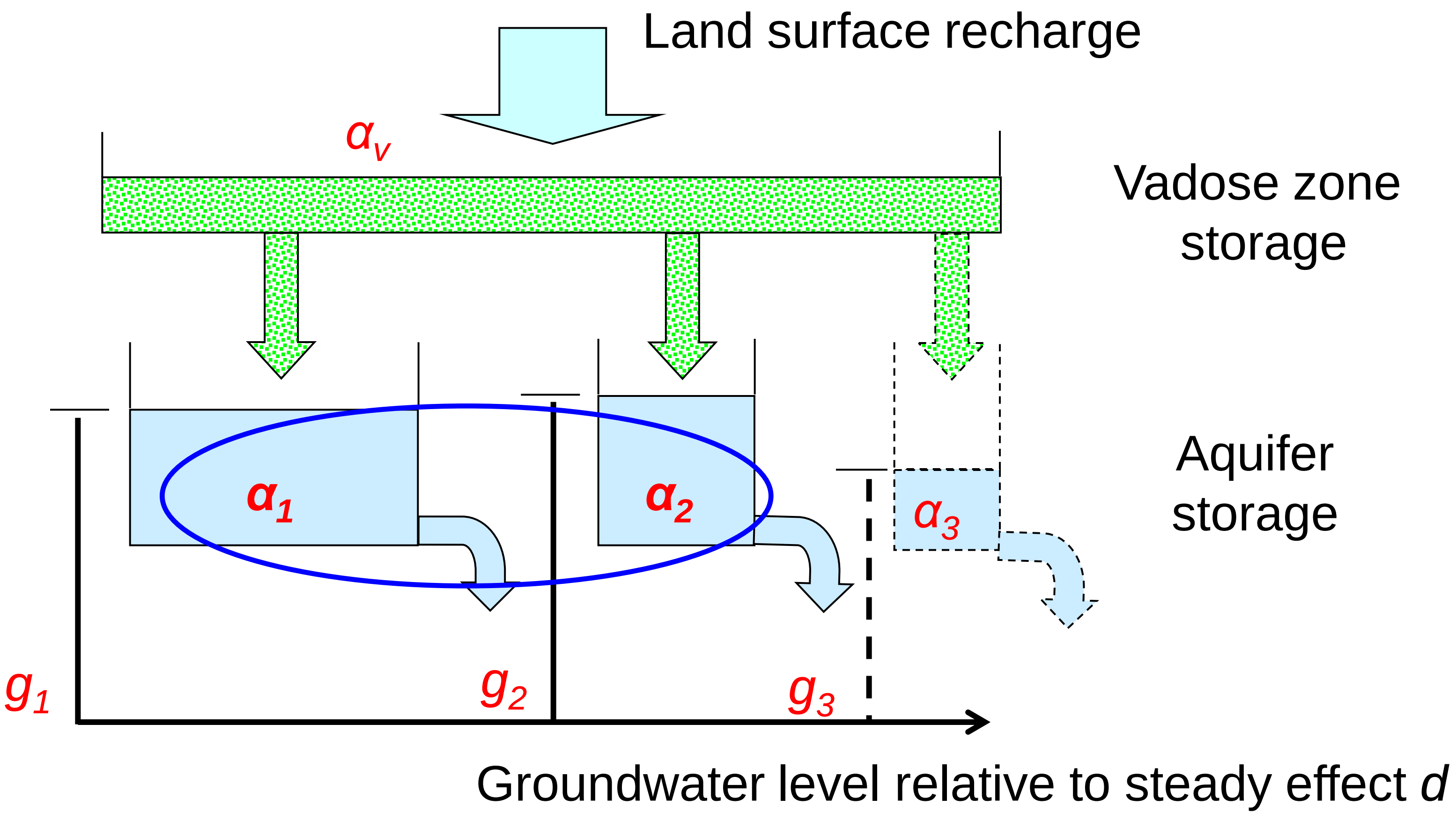


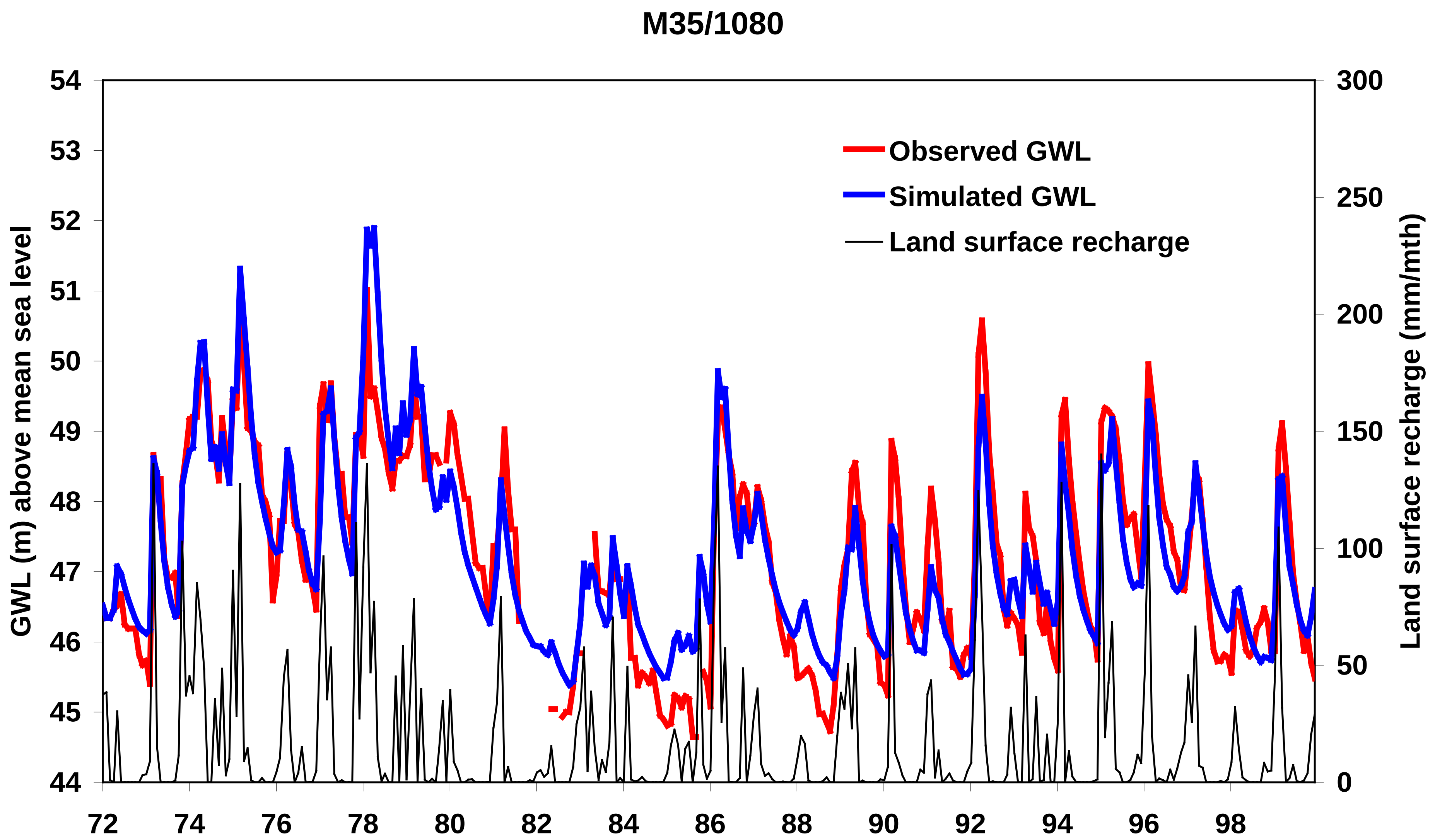

L36/0092
Observed GWL
Simulated GWL
Land surface recharge
GWL (m) above mean sea level
Land surface recharge (mm/mth)
100
95
90
85
80
75
70
65
60
300
250
200
150
100
50
0
72
74
76
78
80
82
84
86
88
90
92
94
96
98

Monitoring wells L36/0092 & M35/1080



Significant eigenvalues for other wells - monthly data - no vadose zone effect – fixed recharge pattern





M35/1080
Observed GWL
Simulated GWL
Land surface recharge
GWL (m) above mean sea level
Land surface recharge (mm/mth)
54
53
52
51
50
49
48
47
46
45
44
300
250
200
150
100
50
0
72
74
76
78
80
82
84
86
88
90
92
94
96
98

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4. **Eigenstructure of numerical groundwater flow model**

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Groundwater flow equation

(with time derivative on LHS)

$$S\frac{\partial h}{\partial t} = \underbrace{\frac{\partial}{\partial x}\left(T_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial h}{\partial y}\right)} + R$$

Discretize the spatial operator

State-space groundwater flow model
- in continuous-time, discrete space

- modified from Kuiper (1973); Andreu and Sahuquillo (1987)

State

$$\frac{d}{dt}\left|h_{i,j}(t)\right| = \left|\frac{T_{x\,i,j}}{S_{i,j}\Delta x_{i,j}^{\;2}}, \frac{T_{y\,i,j}}{S_{i,j}\Delta y_{i,j}^{\;2}}\right|\left|h_{i,j}(t)\right| + \left|\frac{1}{S_{i,j}}\right|^{T}\left|R_{i,j}(t)\right|$$

System matrix **A**
defined by aquifer
properties, scale
& boundaries

Input matrix **B**

Input

State-space groundwater flow model
- in continuous-time, discrete space
(modified from Andreu and Sahuquillo, 1987)

State

$$\frac{d}{dt}\left|h_{i,j}(t)\right| = \left|\frac{T_{x\,i,j}}{S_{i,j}\Delta x_{i,j}^{\;2}}, \frac{T_{y\,i,j}}{S_{i,j}\Delta y_{i,j}^{\;2}}\right|\left|h_{i,j}(t)\right| + \left|\frac{1}{S_{i,j}}\right|^{T}\left|R_{i,j}(t)\right|$$

System matrix **A**
defined by aquifer
properties, scale
& boundaries

Input matrix **B**

Input

System equation: $\dot{\mathbf{h}} = \mathbf{A}\mathbf{h} + \mathbf{B}\mathbf{r}$

Relevance of state-space system description

System equation $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Br}$

Observation equation $\mathbf{y} = \mathbf{Cx} + \mathbf{Dr}$

output state input

- Basis of controllability and observability theory
- Applicable to deterministic and stochastic linear systems
- Useful for quasi-linear or linearised system approximations
- Basis for design of system control

Eigenstructure of the system equation

System equation for states **h**

$$\dot{\mathbf{h}} = \mathbf{A}\mathbf{h} + \mathbf{B}\mathbf{r}$$

Obtain eigenvectors **V**
and eigenvalues $\boldsymbol{\alpha}$ of
system matrix **A**

$$\mathbf{A} = \mathbf{V}\boldsymbol{\alpha}\mathbf{V}^{-1}$$

System equation for states **L**
and observation equation
for states **h**

$$\dot{\mathbf{L}} = \boldsymbol{\alpha}\mathbf{L} + \mathbf{V}^{-1}\mathbf{B}\mathbf{r}$$

$$\mathbf{h} = \mathbf{V}\mathbf{L}$$

Advantages of the eigenstructure system model description

- Model order can be reduced to a few eigenvalues for many applications

- Predictions can be made for any length of stress (input) period without calculations at intermediate time intervals

- Reduced-order models with flexible time period are computationally faster than the equivalent numerical model

- Dynamic predictions can be more accurate than discrete-time numerical models because of computational cell inertia

- Reduced-order dynamic models are suitable for spreadsheet

Some disadvantages

- Computation cost of determining the eigenstructure of large system matrices is justified as a “one-time” task for linear models
- Non-linearities: in boundary conditions, such as drying of springs and streams; or transmissivity varies with changes in saturated aquifer thickness
- Would require frequent recalculation of the eigenstructure
- There are some “tricks” to avoid recalculation – for some non-linear cases

General equations for uniform input period

$$r(x,y,t) = \bar{r}(x,y) \quad \text{during} \ (t - \Delta t, t)$$

$$L_i(t) = \exp(-\alpha_i \Delta t) L(t - \Delta t) + \frac{[1 - \exp(-\alpha_i \Delta t)]}{\alpha_i} \boldsymbol{\beta}_i \bar{r}(x,y)$$

$$\mathbf{h} = \mathbf{V}\mathbf{L} \qquad \mathbf{L} \ \text{may be a reduced set of} \ \mathbf{L}$$

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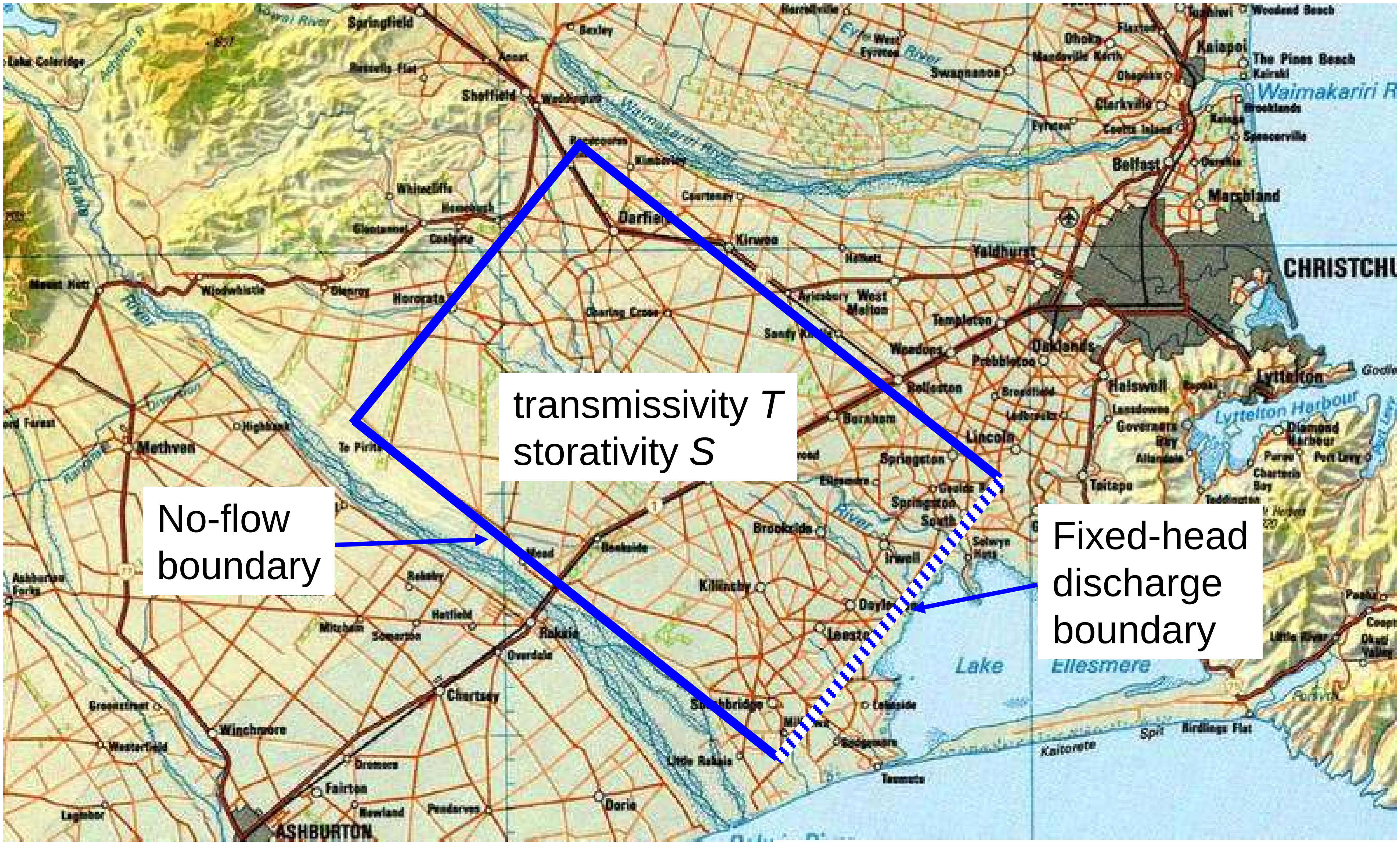
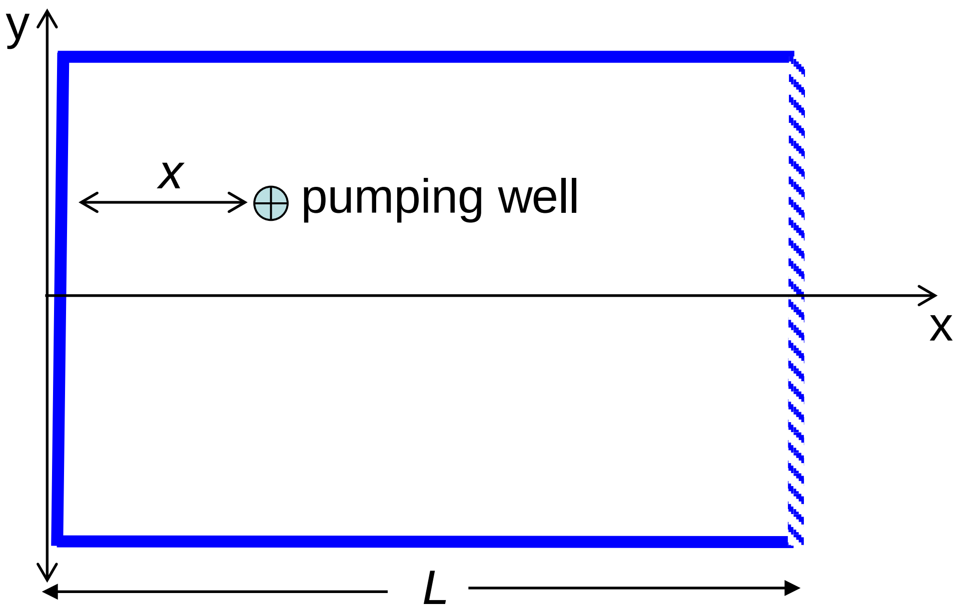
A simplified aquifer



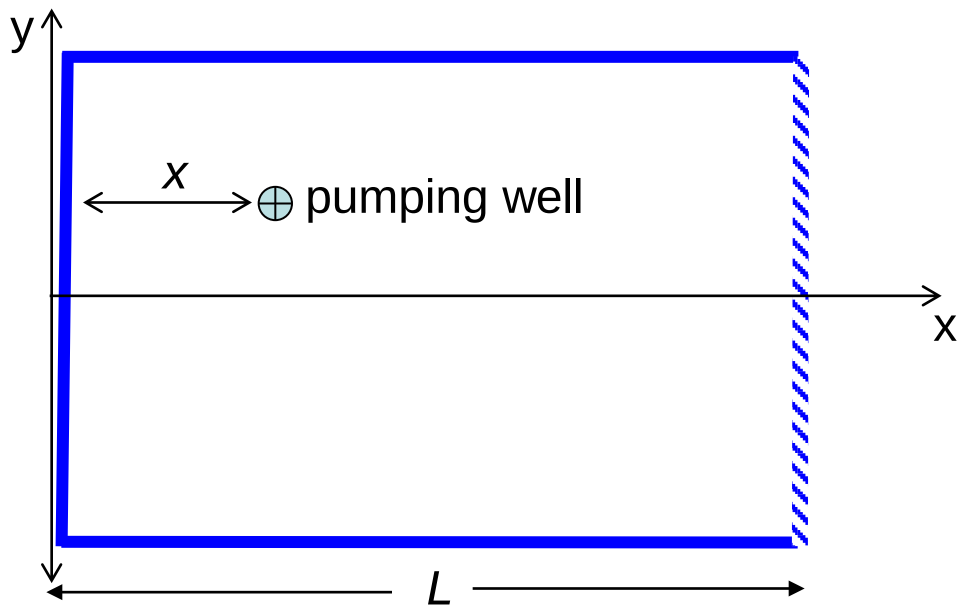
Analytical eigenstructure for the simple aquifer
(Pulido-Velazquez et al., 2005)



- Total pumping effect (stream depletion) at the discharge boundary is a 1D (x-axis) problem.

- Useful for large-scale view of an environmental effect

Analytical eigenstructure for the simple aquifer
(Pulido-Velazquez et al., 2005)



Eigenvalues $\alpha_i = (2i - 1)^2 \dfrac{\pi^2 T}{4SL^2} \quad i = 1,2,...n$

Eigenfunctions $V_i(x) = \dfrac{1}{S} \cos\left[\dfrac{(2i-1)\pi x}{2L}\right]$

Groundwater profile for 1D simple aquifer
(derived from Sloan, 2000)

Groundwater profile described by $h(x,t)$

$$h(x,t) = \sum_{i=1}^{n} L_i(t)$$

$$L_i(t) = \exp(-\alpha_i \Delta t) L(t - \Delta t) + \frac{[1 - \exp(-\alpha_i \Delta t)]}{\alpha_i} \beta_i \bar{r}(t)$$

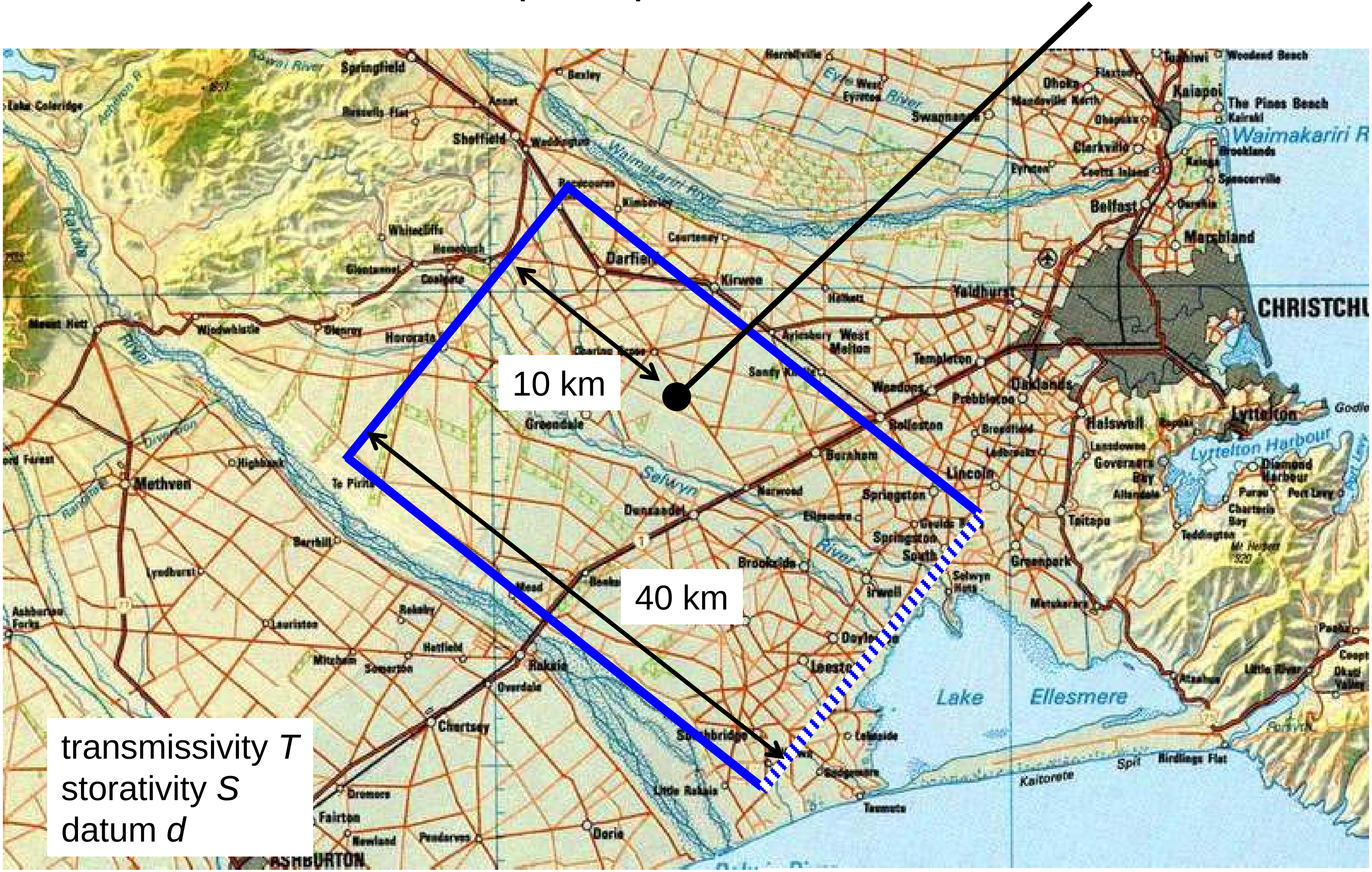
$$\beta_i = \frac{c_i V_i(x)}{\alpha_i}$$

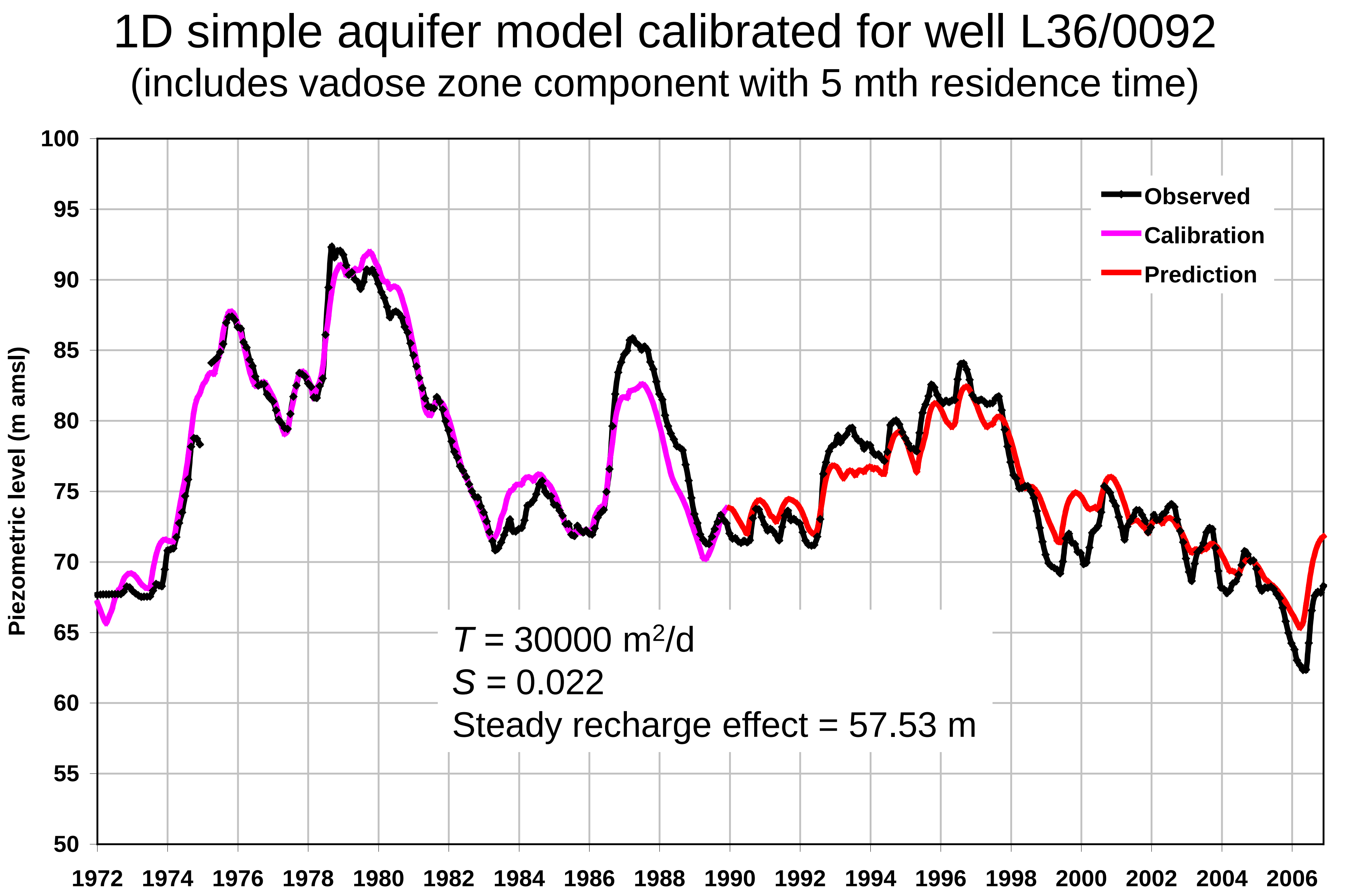
$$c_i = \frac{4(-1)^{i+1}}{(2i-1)\pi}$$

Eigenvalues $\alpha_i = (2i-1)^2 \dfrac{\pi^2 T}{4SL^2} \qquad i = 1,2,\ldots n$

Eigenfunctions $V_i(x) = \dfrac{1}{S} \cos\left[\dfrac{(2i-1)\pi x}{2L}\right]$

Calibrate 1D simple aquifer model to well L36/0092





1D simple aquifer model calibrated for well L36/0092
(includes vadose zone component with 5 mth residence time)
Piezometric level (m amsl)
100
95
90
85
80
75
70
65
60
55
50
1972
1974
1976
1978
1980
1982
1984
1986
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
Observed
Calibration
Prediction
T = 30000 m2/d
S = 0.022
Steady recharge effect = 57.53 m

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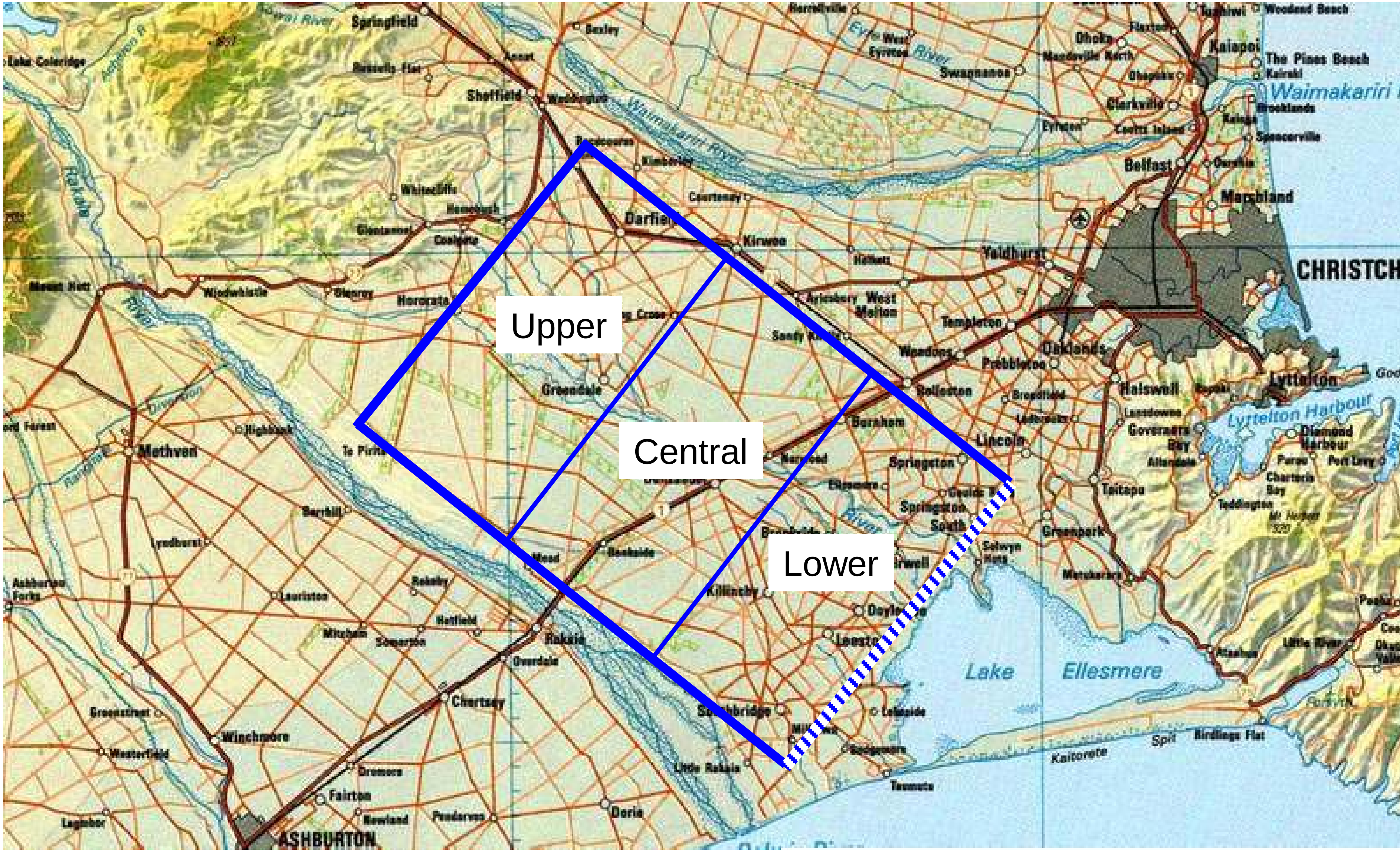
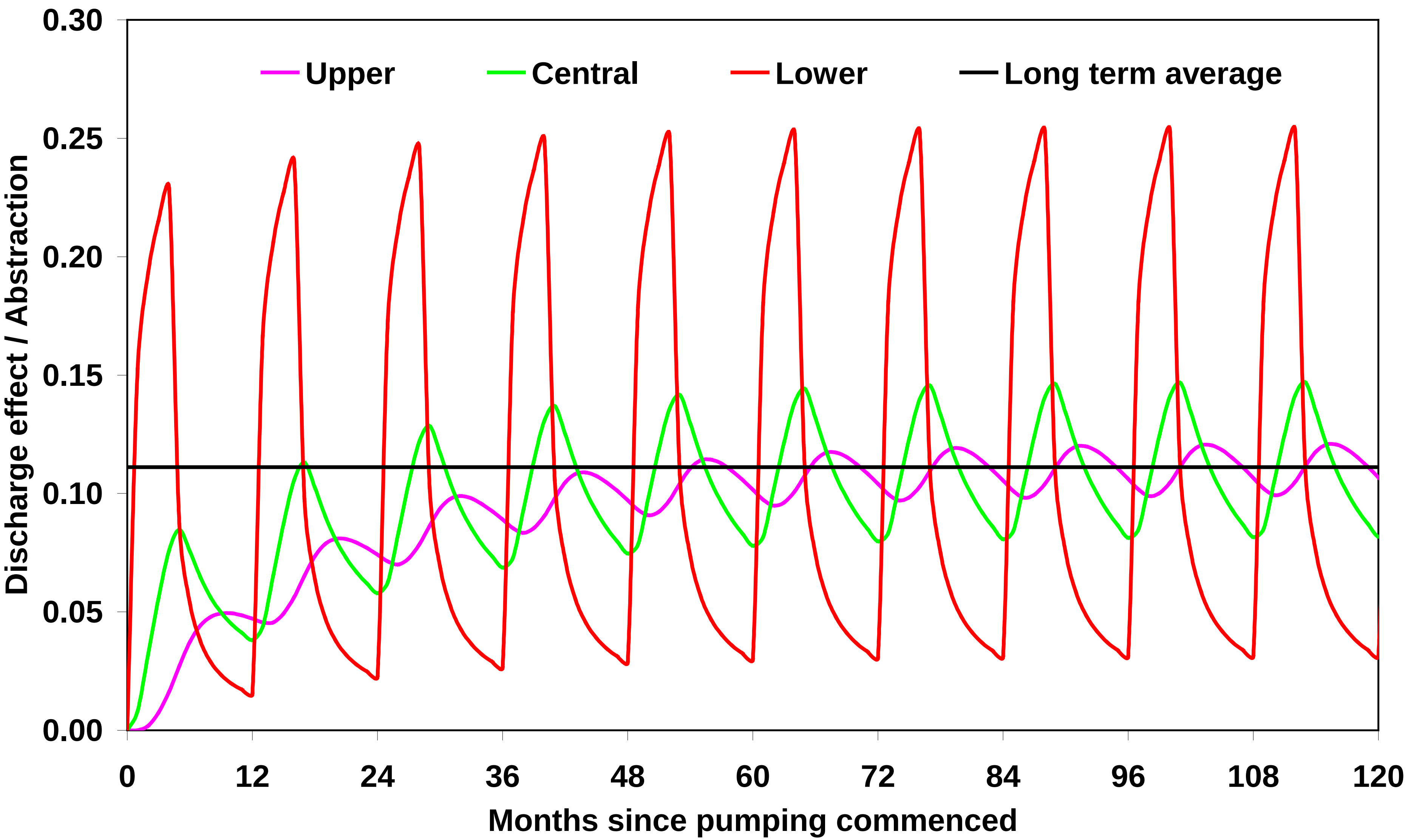
Irrigation pumping abstraction zones



Effect on groundwater discharge of irrigation pumping at unit rate in each zone for four months of every year



Issues for aquifer management

- Intra-seasonal control of lower zone irrigation pumping gives immediate action during low stream flows but high risk for farmers
- Upper zone pumping has a steady effect on streamflow, which lowers the threshold for all users
- Upper zone effect takes years to be fully developed
- Groundwater in each zone appears to have different degrees of reliability
- Control actions must fit with agricultural decision making

The role of simple eigenmodels in aquifer management

- Theoretical basis in physics of groundwater flow
- Equivalence to reduced forms of numerical groundwater models
- Parsimonious parameter set is easily calibrated
- Describe dynamic behaviour of groundwater in form suitable for applications of control theory
- Models as difference equations for deterministic and stochastic (forecasting) applications
- Suitable for spreadsheet applications of models for public education and discussion

Some implementation milestones

- 2003: The “Eigenmodel” approach to analysis of groundwater level data, as web-accessible manual (spreadsheet models on request)
- 2007: Evidence about cumulative effects of irrigation abstraction for major Canterbury water allocation consent hearing
- 2008: Evidence about groundwater mounding under proposed Canterbury irrigation scheme, at major resource consent hearing
- 2009: Technical report on proposals for adaptive management of groundwater produced by Canterbury Regional Council
- Increasing appreciation by irrigation farmers of the dynamics of effects from groundwater abstraction